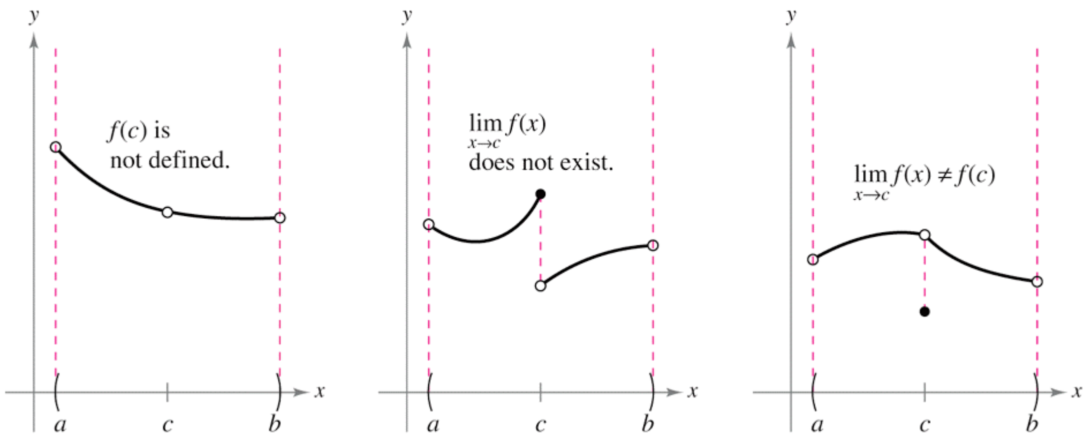
Math 150, *Lecture Notes- Bonds* Name____________

Section 1.4 *Continuity and One-Sided Limits*

Continuity at a Point and on an Open Interval



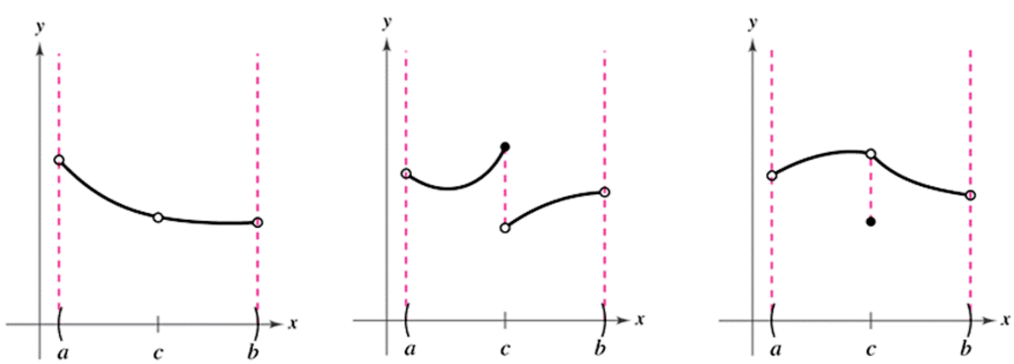
Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

Definition of Continuity

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**



(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

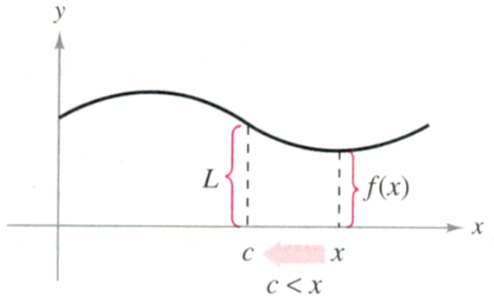
Discuss the continuity of each function.

Ex.1 $f(x)=\dfrac{1}{x^2-9}$

Ex.2 $g(x)=\dfrac{x^2-4x-5}{x+1}$

Ex.3 $h(\theta)=\csc(\theta)$

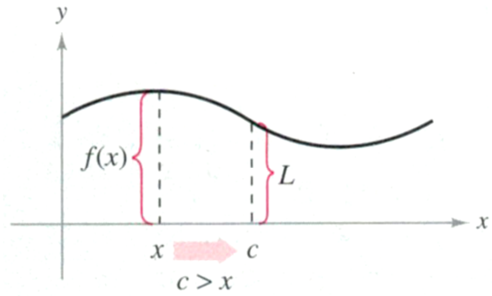
One-Sided Limits and Continuity on a Closed Interval



(a) Limit as x approaches c from the right.

$$\lim_{x \to c^+} f(x) = L.$$

limit from the right



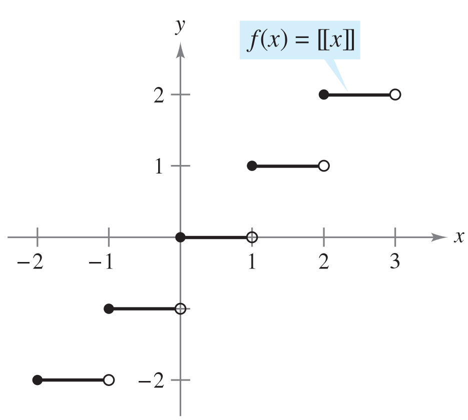
(b) Limit as x approaches c from the left.

$$\lim_{x \to c^-} f(x) = L.$$

limit from the left

Ex.4 $$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

Ex.5 $$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x.$$



Greatest integer function

When the limit from the left is not equal to the limit from the right, the (two-sided) ***limit does not exist***.

THEOREM 1.10 The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

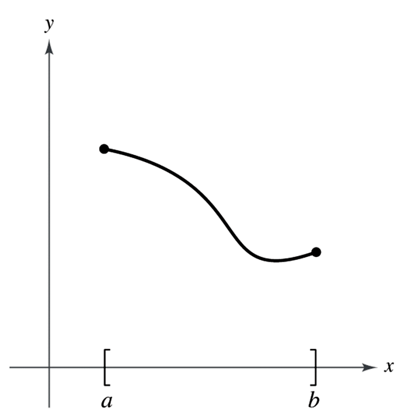
$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).



Ex.6 $f(x) = \sqrt{9 - x^2}$

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
2. Rational: $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Ex.7 Describe the intervals on which the following functions are continuous.

(a) $f(x) = \frac{x+1}{\sqrt{x}}$

(b) $g(x) = x\sqrt{x+3}$

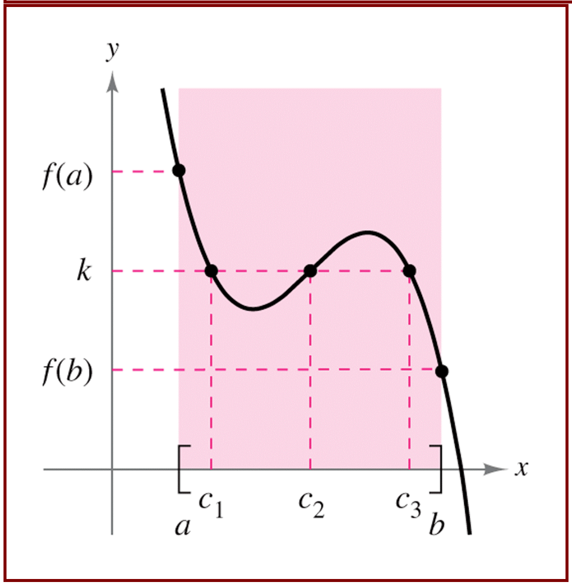
(c) $h(x) = \cos\left(\dfrac{1}{x}\right)$

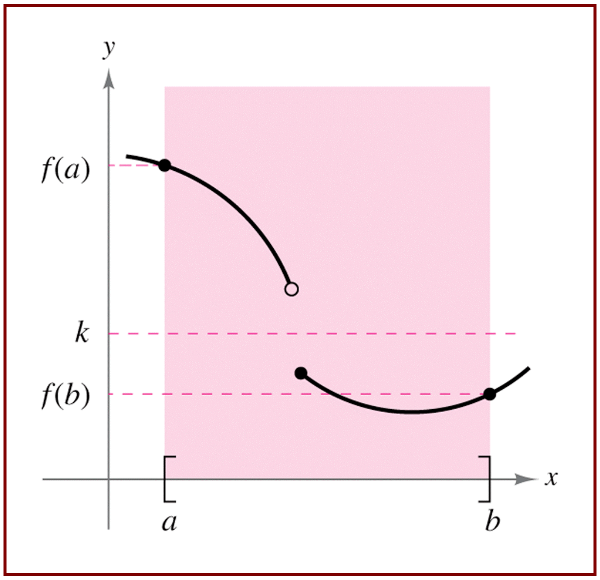
(d) $f(x) = \begin{cases} 2x-4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$





Ex.8 Verify that the Intermediate Value Theorem applies to $f(x) = x^2 - 6x + 8$ on $[0,3]$, and then the value of c guaranteed by the theorem, where $f(c) = 0$.

Ex.9 Find the constant a such that $f(x)=\begin{cases} 3x^3, & x\le 1 \\ ax+5, & x>1 \end{cases}$ is continuous on the entire real number line.